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Asymptotic Analysis of the Boltzmann Equation for Dark Matter Relics

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This paper presents an asymptotic analysis of the Boltzmann equations (Riccati differential equations) that describe the physics of thermal dark-matter-relic abundances. Two different asymptotic techniques are used, boundary-layer theory, which makes use of asymptotic matching, and the delta expansion, which is a powerful technique for solving nonlinear differential equations. Two different Boltzmann equations are considered. The first is derived from general relativistic considerations and the second arises in dilatonic string cosmology. The global asymptotic analysis presented here is used to find the long-time behavior of the solutions to these equations. In the first case the nature of the so-called freeze-out region and the post-freeze-out behavior is explored. In the second case the effect of the dilaton on cold dark-matter abundances is calculated and it is shown that there is a large-time power-law fall off of the dark-matter abundance. Corrections to the power-law behavior are also calculated.

I. INTRODUCTION

The thermal history of nonbaryonic dark-matter (DM) species is highly relevant to the shaping of the universe as we find it today. The existence of DM is based on evidence at many length scales. At the scale of galactic halos, for example, DM explains the observed flatness of the rotation curves of spiral galaxies [1]. According to observations over the past twelve years, 23% of the energy of the universe consists of DM. This number has been obtained by best-fit analyses of astrophysical data to the Standard Cosmological Model, which is a Friedmann-Robertson-Walker cosmology involving cold DM as the dominant DM species. The modern data is based on observations of type-Ia supernovae [2], the cosmic microwave background [3, 4], baryon oscillations [5], and weak-lensing data [6]. It should be stressed that estimates of the DM abundance depend crucially on the theoretical model that is considered.

In the absence of dilaton effects from string theory, the evolution of the appropriately normalized number density $Y(x)$ of a DM species X of mass m_X is governed by the Boltzmann equation

$$Y'(x) = -\lambda x^{-n-2}\left[Y^2(x) - Y_{\rm eq}^2(x)\right], \tag{1}$$

which is a Riccati equation in the dimensionless independent variable $x \equiv m_X/T$, where T is the temperature. The parameter λ is a dimensionless measure of the scattering of DM particles and is regarded as a large number $\lambda \gg 1$. The integer $n = 0, 1, 2, \ldots$ comes from a partial-wave analysis of the scattering of DM particles; $n = 0$ refers to S-wave scattering. For bosonic remnants the function $Y_{\rm eq}(x)$ is the distribution [7]

$$Y_{\rm eq}(x) = A\int_0^\infty ds\frac{s^2}{e^{\sqrt{s^2+x^2}}-1}, \tag{2}$$

where $A = 0.145g/g_*$, g is the degeneracy factor for the DM species, and g_* counts the total number of massless degrees of freedom [8].

As the universe cools and x increases, the nature of the solution $Y(x)$ to (1) changes rapidly in the vicinity of a value $x = x_f$, the so-called *freeze-out* point, and as $x \to \infty$ the solution $Y(x)$ approaches the constant Y_∞, called the *relic abundance*. Because a closed-form analytical solution to this Riccati equation is unavailable, an approximate heuristic approach is customarily used to treat this Riccati equation: One approximation is made for $x < x_f$ and another is made for $x > x_f$. The solutions in the two regions are then patched at $x = x_f$. This approach gives an intuitive and reasonably accurate determination of Y_∞ and it is widely adopted [8].

However, this splitting into two regions is only a mathematical convenience and there is really no precise value x_f. Because the differential equation (1) is first order, its solution is completely determined by one initial condition,

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namely $Y(0)$. The usual method of splitting (1) into two approximate first-order equations, which are valid in each of two regions, leads to two conditions, an initial condition and a patching condition. We feel that this gives rise to an unsatisfactory mathematical discussion that is prevalent in the literature. The value of x_f, for example, becomes explicitly involved in the determination of Y_∞ when there is no reason for this.

Equation (1) is valid in a general-relativistic framework. However, given the importance of understanding the current thermal-relic abundance of DM in theories beyond the standard model of particle physics, we also reexamine here the modifications of (1) due to string cosmology [9]. String theory is widely accepted as a leading candidate for physics beyond the standard model, and it places a constraint on the types of time-dependent backgrounds in conformally invariant critical theories. As before, we are interested in eras in which the temperature T satisfies $m_X > T > T_0$, where T_0 is the current temperature of the universe. String cosmology leads to a rolling dilaton source in the Boltzmann equation [10] that describes DM species. Including this source gives an additional linear term in the Boltzmann equation:

$$Y'(x) = -\lambda x^{-n-2}\left[Y^2(x) - Y_{\rm eq}^2(x)\right] + \Phi_0 Y(x)/x, \tag{3}$$

where Φ_0 is a negative dimensionless constant of order 1.

The purpose of this paper is to study analytically the two Riccati equations (1) and (3). These equations do not have exact closed-form solutions. However, because λ is a large parameter, one can attempt to find asymptotic approximations to the solutions. The most direct approach is to convert these Riccati equations into equations of Schrödinger type. When this transformation is applied to (1), we obtain

$$v''(x) - \frac{n(n+2)}{4x^2}v(x) - \lambda^2 x^{-2n-4}Y_{\rm eq}^2(x)v = 0. \tag{4}$$

Now, if we set $n=0$, we obtain the standard time-independent Schrödinger equation in which $1/\lambda$ plays the role of $\hbar$. While it is possible to perform a local analysis of this equation for small x and for large x, it is not easy to use WKB analysis to find a global asymptotic approximation because the equation is singular at $x=0$ and there is a turning point at $x=\infty$.

Thus, in this paper we will use two other powerful asymptotic methods from which we can extract global information. The first method is boundary-layer analysis. This asymptotic technique, which has been used to solve approximately the equations of fluid mechanics, gives very accurate results, and it has the physical advantage of treating freeze-out as a boundary-layer region, very much like the boundary between two fluids. The second technique, known as the *delta expansion* [11], is particularly well-suited to study the transition from the equilibrium region to the large-x behavior of the solutions without the necessity of finding approximations to the Boltzmann equation in different epochs. We will see that the presence of a source in (3) gives a solution for $Y(x)$ in (3), whose qualitative behavior is significantly different from the solution for $Y(x)$ in (1).

This paper is organized as follows. In Sec. II we summarize the derivation of the Boltzmann equations (1) and (3). In Sec. III we apply boundary-layer analysis to study (1) and (3). Next, in Sec. IV we describe the delta expansion and then use it to study the approximate behaviors of (1) and (3). Finally, in Sec. V we give some brief concluding remarks.

II. DERIVATION OF THE BOLTZMANN EQUATIONS

In this section we review the derivation of the two Boltzmann equations (1) and (3).

A. Derivation of (1)

In the hot early universe DM particles interact with themselves and with other particles. Particle species are assumed to react rapidly enough to maintain equilibrium. However, the universe expands and cools throughout its history. The timescale associated with this expansion is determined by the Hubble rate H. There is also a timescale Γ associated with the scattering cross-section (that is, an interaction rate per particle). The dynamics of DM particles depends on the ratio Γ/H. When $\Gamma/H \gg 1$, conditions for equilibrium hold and $Y(x)$ follows the canonical distribution obtained from equilibrium statistical mechanics. However, for $\Gamma/H \ll 1$ the DM particles are unable to maintain equilibrium. There is a crossover to freeze-out behavior in which $Y(x)$ is asymptotically a constant.

Let us consider a two-body scattering process in which particles of species 1 and 2 scatter reversibly into particles of species 3 and 4. The phase-space distribution function $f_i\left(\vec{r}, \vec{p}, t\right)$ for the species i gives the number of particles

in an infinitesimal region of phase space around the position $\vec{r}$ and momentum $\vec{p}$: $f_i(\vec{r},\vec{p},t)\, d^3r\, d^3p$. The main bulk quantity of interest is the number density $n_i(\vec{r},t)$, which is given by [8]

$$n_i(\vec{r},t) = g_i \int \frac{d^3p}{(2\pi)^3}\, f_i(\vec{r},\vec{p},t)\,, \tag{5}$$

where g_i is the degeneracy factor for the ith DM species. The evolution of such a bulk quantity in the universe is given by the Liouville equation (in the absence of collisions, for simplicity)

$$\frac{df_i}{dt} = L[f_i] \equiv \left(\frac{\partial}{\partial t} + \frac{d\vec{p}}{dt}\cdot\nabla_{\vec{p}} + \frac{d\vec{r}}{dt}\cdot\nabla_{\vec{r}}\right) f_i = 0. \tag{6}$$

The standard Robertson-Walker metric for an isotropic and expanding flat universe is given by

$$ds^2 = -dt^2 + a^2(t)\left(dx^2+dy^2+dz^2\right), \tag{7}$$

where $a(t)$ is the scale factor [12]. The covariant generalization of (6) is [8],

$$L\left[f_i\right] = \left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \frac{\partial}{\partial p^\mu}\right) f_i = 0, \tag{8}$$

where the Christoffel symbol is given by

$$\Gamma^\mu_{\nu\rho} \equiv g^{\alpha\mu}\left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}\right)/2.$$

For the metric in (7), isotropy further implies that $f_i(\vec{p},\, t) = f_i(|\vec{p}|\,, t)$. For the isotropic case (8) takes the form

$$L[f(E,t)] = E\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}\,|\vec{p}|^2\,\frac{\partial f}{\partial E},$$

where $E = \sqrt{\vec{p}^2 + m^2}$.

For a two-body collision process the Liouville equation (8) no longer has a vanishing right side. This equation can then be used to describe the change in the number density of a given species. For species 1, for example, one gets [8]

$$\begin{aligned} a^{-3}\frac{d\left(n_1 a^3\right)}{dt} &= \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \\ &\quad \times (2\pi)^4 \delta^3\left(p_1+p_2-p_3-p_4\right)\delta\left(E_1+E_2-E_3-E_4\right)|\mathcal{A}|^2 \\ &\quad \times \left[f_3 f_4 \left(1\pm f_1\right)\left(1\pm f_2\right) - f_1 f_2\left(1\pm f_3\right)\left(1\pm f_4\right)\right], \end{aligned} \tag{9}$$

where the plus sign is used for a bosonic species and the minus sign is used for a fermionic species. The symbol $\mathcal{A}$ represents the scattering amplitude for the process $1+2 \leftrightarrow 3+4$ and it is a function of the p_i.

If the scattering process is sufficiently fast, f_i can be parametrized by canonical Fermi-Dirac or Bose-Einstein distributions. For temperatures $T \ll E - \mu$ the Bose-Einstein and Fermi-Dirac distributions both take the form

$$f(E) \sim e^{\mu/T} e^{-E/T}, \tag{10}$$

which implies that quantum statistics are not important. Hence, the Pauli-blocking and Bose-enhancement are negligible ($f_i \ll 1$), and the third line of (9) simplifies:

$$f_3 f_4\left(1\pm f_1\right)\left(1\pm f_2\right) - f_1 f_2\left(1\pm f_3\right)\left(1\pm f_4\right) \sim e^{-(E_1+E_2)/T}\left[e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T}\right],$$

where the relation $E_1 + E_2 = E_3 + E_4$ has been used. Also, combining (5) and (10), we get

$$n_i = g_i e^{\mu_i/T} \int \frac{d^3p}{(2\pi)^3} e^{-E_i/T}.$$

The equilibrium number density in the absence of a chemical potential is denoted by $n_i^{(0)}$. Thus,

$$a^{-3}\frac{d}{dt}\left(n_1 a^3\right) = n_1^{(0)} n_2^{(0)} \langle\sigma v\rangle \left\{\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right\}, \tag{11}$$

where the thermally averaged annihilation cross-section $\langle \sigma v \rangle$ is given by

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T} \times (2\pi)^4 \delta^3 (p_1 + p_2 - p_3 - p_4)\, \delta (E_1 + E_2 - E_3 - E_4)\, |\mathcal{A}|^2. \tag{12}$$

We now make the standard assumption [8] that the predominant interaction of the cold DM species X of mass m_X is $XX \leftrightarrow ll$, where l is a light particle in equilibrium. As a consequence, in (11) we can replace n_1 and n_2 by n_X, where n_X is the number density of the species X. Also, we replace and n_3 and n_4 by $n_l^{(0)}$. The resulting equation is

$$a^{-3} \frac{d}{dt} \left(n_X a^3 \right) = \langle \sigma v \rangle \left[\left(n_X^{(0)} \right)^2 - n_X^2 \right]. \tag{13}$$

We now define $x \equiv m/T$ and note that $dx/x = -dT/T = da/a$ because T scales as $1/a$. Thus, $\frac{dx}{dt} = Hx$, where the Hubble rate $H \equiv \frac{d}{dt} \log(a)$. Since the cosmological era for DM production is radiation dominated, $a(t) \propto \sqrt{t}$. This translates into $H = H_m/x^2$ with $H_m = 1.67 g_*^{1/2} m_X^2 / m_{\text{Planck}}$. It is known theoretically [8] that $\sigma v \propto v^{2n}$ with $n = 0$ for s-wave annihilation and $n = 1$ for p-wave annihilation. Since $\langle v \rangle \propto \sqrt{T}$, we have the parametrization $\langle \sigma v \rangle = \sigma_0 x^{-n}$ for $x \geq 3$ [8].

Finally, we introduce the dependent variable $Y \equiv n_X/T^3 \propto n_X a^3$. Similarly, we define $Y_{\text{eq}} \equiv n_X^{(0)}/T^3$. We then obtain the Boltzmann equation in (1), where $\lambda \equiv \sigma_0 m_X^3 / H_m \propto m_{\text{Planck}}/m_X$, and this explains why λ is a large dimensionless parameter [13].

B. Derivation of (3)

String theory can be formulated in nonflat backgrounds, which is necessary when considering cosmology. Here, we consider the world-sheet sigma-model approach for dilaton-based cosmologies [9]. In superstring theory the bosonic part of the supermultiplet with lowest energy consists of the following massless states: the graviton g_{MN}, the spinless dilaton Φ, and the antisymmetric spin-one tensor B_{MN}. For expanding universes Φ provides consistent time-dependent backgrounds. In such backgrounds the string sigma model on the world sheet Σ is given by [14]

$$S_\sigma = \int_\Sigma \frac{d^2\sigma}{4\pi\alpha'} \left[\sqrt{\gamma}\, \gamma^{\alpha\beta} g_{MN}(X) \partial_\alpha X^M \partial_\beta X^N + B_{MN}(X) \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N + \alpha' \sqrt{\gamma} \Phi(X) R^{(2)}/2 \right], \tag{14}$$

where X^M are target space-time coordinates with $M, N = 0, 1, \ldots, 9$, σ^α are the world-sheet coordinates with $\alpha, \beta = 0, 1$, $\gamma^{\alpha\beta}$ is the world-sheet metric, $\gamma = \left| \det \left(\gamma^{\alpha\beta} \right) \right|$, $R^{(2)}$ is the Ricci scalar associated with $\gamma^{\alpha\beta}$, and α' is the string slope. Expanding around a conformal flat background with the action S^*, we can write S_σ as

$$S_\sigma = S^* + h^i \int_\Sigma d^2\sigma\, V_i, \tag{15}$$

where h^i denotes the background fields $\{g_{MN}, B_{MN}, \Phi\}$ and V_i are associated vertex operators [14].

Short-distance singularities of the quantum field theory on the world sheet lead to renormalized couplings $\left\{ h_R^i \right\}$ and to dependence on the renormalization-group scale μ [15]. Usually, this results in nonvanishing β functions: $\beta^i \equiv dh_R^i / d\log\mu$. To restore conformal invariance, these β functions must vanish. This leads to equations of motion satisfied by the background fields. The usual procedure is to consider an effective target-space action in the string frame that reproduces the equations of motion:

$$S = -\frac{1}{2\alpha'^4} \int d^{10}x \sqrt{G}\, e^{-\Phi} \left[R + (\nabla\Phi)^2 + 2\alpha'^4 U(\Phi) - \tilde{H}^2/12 \right], \tag{16}$$

where $\tilde{H}^2 = H_{\mu\nu\alpha} H^{\mu\nu\alpha}$, $H_{\mu\nu\alpha} \equiv \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu}$, and the potential $U(\Phi)$ has been introduced. With the help of duality symmetries it is possible to find analytic solutions for the time dependence of the dilaton field [9].

From (16) it can be shown [10] that in three spatial dimensions the energy density ρ of the DM species X satisfies

$$\frac{d\rho}{dt} + 3H(\rho + p) - \frac{d\Phi}{dt}(\rho - 3p) = 0. \tag{17}$$

We then assume that the thermal DM species X behaves like dust (that is, $p = 0$) and that the energy density of the DM is given by the simple formula $\rho = m_X n_X$. Next, in place of the 0 on the right side of (17), we include a collision term, which is just the right side of (13):

$$\frac{d}{dt}n_X + 3Hn_X - \frac{d\Phi}{dt}n_X = \langle\sigma v\rangle \left[\left(n_X^{(0)}\right)^2 - n_X^2\right]. \tag{18}$$

Assuming that matter sources are perfect fluids and requiring scale-factor duality symmetry, one can show [9] that up to an additive constant, $\Phi(t) = \Phi_0 \log a(t)$ where $\Phi_0 = \mathrm{O}(1)$ and $\Phi_0 < 0$.

Finally, we make the assumption that the behavior of the DM species is dominated by radiation so that $a(t) \propto t^{1/2}$ [8]. As in Subsec. II A, we introduce the variables $Y(x)$ and x and obtain the Boltzmann equation (3).

III. BOUNDARY-LAYER SOLUTION TO (1) AND (3)

In this section we show how to perform a boundary-layer asymptotic analysis of (1) and (3). The advantage of this analysis is that it provides a global picture of the cosmological development from the initial time to the present as described by the Boltzmann equation, and not just the physics of the equilibrium epoch or of the post-equilibrium epoch alone. It also establishes a framework to describe in a clear and natural way the region of rapid transition between these two epochs. A satisfactory description of this crucial transition is lacking in earlier treatments in the literature because the earlier analysis used *patching* (joining together two solutions to a differential equation at an arbitrary and fictitious point, which produces an elbow in the solution) rather than *asymptotic matching* [16].

One may wonder why an asymptotic procedure as powerful as boundary-layer theory should be used to solve a first-order ordinary differential equation as simple as a Riccati equation. A general Riccati equation

$$y'(x) = a(x)y^2(x) + b(x)y(x) + c(x)$$

can be recast as a linear second-order equation,

$$w''(x) - \left[\frac{a'(x)}{a(x)} + b(x)\right] w'(x) + a(x)c(x)w(x) = 0, \tag{19}$$

where $y(x) = -\frac{w'(x)}{a(x)w(x)}$. Furthermore, (19) can be recast as a Schrödinger equation

$$v''(x) + \left\{p''(x)/p(x) - [b(x) + a'(x)/a(x)]^2/2 + a(x)c(x)\right\} v(x) = 0 \tag{20}$$

by introducing $v(x) = w(x)/p(x)$, where $p'(x)/p(x) = [b(x) + a'(x)/a(x)]/2$.

The form (20) is often useful for asymptotic WKB analysis but the problem of freeze-out poses mathematical difficulties. If we apply these transformations to (1) for the case $n = 0$ and use the leading asymptotic forms for $Y_{\rm eq}(x)$ in Appendices A and B, we obtain the Schrödinger equations

$$v''(x) - \lambda^2 x^{-4}\eta^2 v(x) = 0, \tag{21}$$

where $\eta \equiv 2A\zeta(3)$ for $x \ll 1$, and

$$v''(x) - \lambda^2 A^2 x^{-1} e^{-2x} v(x) = 0 \tag{22}$$

for $x \gg 1$. The role of $\hbar$ in these equations is played by $1/\lambda$ because λ is treated as a large parameter.

The exact general solution of (21) is

$$v(x) = x\left(v_+ e^{\lambda\eta/x} + v_- e^{-\lambda\eta/x}\right), \tag{23}$$

where $v_+$and v_- are constants. The approximate general solution to (22) can be found by using a standard application of WKB [16]. [A detailed analysis of (22) for large x is given in Appendix C.] However, because (22) has a turning point at $x = \infty$ and because (21) has a singularity at $x = 0$, it is very difficult to construct a uniform asymptotic expansion that is valid for all x. We show below that boundary-layer theory overcomes these difficulties.

A. Boundary-layer analysis of (1)

Whenever the highest-derivative term in a differential equation is multiplied by a small parameter, one can attempt a boundary-layer analysis [16]. In such an analysis one identifies an *outer* region (or regions) in which the solution is slowly varying and an *inner* or *boundary-layer* region (or regions) in which the solution is rapidly varying. If these regions have an overlap, one tries to construct a global asymptotic approximation to the differential equation by performing an asymptotic match of the outer solutions to the inner solutions.

In boundary-layer form the derivative term in (1) is multiplied by $1/\lambda$, which is regarded as small ($1/\lambda \ll 1$). Thus, we begin by looking for an outer solution; that is, a solution whose derivative is not large. To leading order such a solution in the outer region satisfies a *distinguished limit* (an asymptotic balance between two of the three terms in the differential equation) in which we neglect the derivative term as $\lambda \to \infty$:

$$Y(x) \sim Y_{\rm eq}(x) \quad (\lambda \to \infty). \tag{24}$$

Since $Y(x)$ is well approximated by $Y_{\rm eq}(x)$ in this region, we call this outer region the *thermal-equilibrium* region.

To higher order, we seek a series expansion of this thermal-equilibrium outer solution as a formal power series in inverse powers of λ:

$$Y^{\rm thermal-equilibrium}(x) \sim \sum_{k=0}^{\infty} \lambda^{-k} Y_k^{\rm thermal-equilibrium}(x). \tag{25}$$

Substituting this series into (1) and collecting powers of $1/\lambda$ yields the higher-order terms in the outer series. For example, to first order we get

$$Y_1^{\rm thermal-equilibrium}(x) = -\frac{1}{2} x^{n+2} \frac{d}{dx} \log\left[Y_{\rm eq}(x)\right]. \tag{26}$$

We must now determine the extent of the thermal-equilibrium region. We know from Appendix A that for large x, $x \gg 1$, the asymptotic behavior of $Y_{\rm eq}(x)$ is given by

$$Y_{\rm eq}(x) \sim A e^{-x} x^{3/2} \qquad (x \to \infty). \tag{27}$$

Thus, for large x in the outer region

$$Y_0^{\rm thermal-equilibrium}(x) \sim A e^{-x} x^{3/2} \quad \text{and} \quad Y_1^{\rm thermal-equilibrium}(x) \sim \frac{1}{2} x^{n+2}. \tag{28}$$

Hence, the second term in the outer series is no longer small compared with the first term when

$$x \sim \log(2A\lambda) - (n+1/2)\log(x). \tag{29}$$

We will call the solution to this asymptotic relation the so-called *freeze-out* value $x_{\rm f}$:

$$x_{\rm f} \sim \log(2A\lambda) - (n+1/2)\log\left(x_{\rm f}\right). \tag{30}$$

Note that if we take $\lambda \approx 10^{14}$ and $A \approx 0.00145$, we see that the outer asymptotic approximation ceases to be valid when x exceeds the approximate numerical value

$$x_{\rm f} \approx 25. \tag{31}$$

Equation (29) defines the upper asymptotic limit of the thermal-equilibrium region. However, it is important to emphasize here that *freeze-out does not occur at a point*; $x_{\rm f}$ should not be viewed as a number but rather as a large range of values of x all satisfying the asymptotic relation (29):

$$x \sim x_{\rm f} \quad (\lambda \to \infty). \tag{32}$$

A second possible distinguished limit of (1) could in principle consist of an asymptotic balance between the left side and the second term on the right side. However, this distinguished limit is inconsistent and must be rejected because we are led to a contradiction: If we solve the resulting equation, we find that for large λ the first term on the right side is in fact *not negligible* compared with the second term .

A third distinguished limit of (1) occurs when x is so large that the contribution of the equilibrium term $Y_{\rm eq}^2(x)$ is negligible. In this case, the left side is asymptotic to the first term on the right side:

$$Y'(x) \sim -\lambda x^{-n-2} Y^2(x) \qquad (x \gg 1). \tag{33}$$

In this second outer region, which we will call the *post-freeze-out* region, the solution $Y^{\rm post-freeze-out}(x)$ to (33) is

$$Y^{\rm post-freeze-out}(x) \sim \frac{1}{1/C - \lambda x^{-n-1}/(n+1)}, \tag{34}$$

where C is a constant of integration to be determined. Note that this solution is consistent and valid when $x \gg 1$ because $Y_{\rm eq}(x)$ is exponentially small when $x \gg 1$. Note also that as $x \to \infty$, $Y^{\rm post-freeze-out}(x)$ approaches the limiting value C. Thus, C represents the long-time limiting value of the relic abundance.

The physical process of freeze-out can be recast in mathematical terms as a process that occurs in an inner region (or boundary layer), which we treat as a time interval that is comparatively short relative to the time intervals of the two outer regions, the thermal-equilibrium region and the post-freeze-out region. We begin the analysis of the freeze-out boundary layer by determining the size of this region. To do so, we introduce the *inner variable* X:

$$x = x_{\rm f} + \kappa X. \tag{35}$$

We regard $|X|$ as a variable that may get large compared to 1, say as large as $X_{\rm max}$, but X is still small compared with λ. Thus, since κ is expected to be a small parameter roughly of order $1/\lambda$, the boundary layer is narrow because it extends roughly from $x_{\rm f} - \kappa X_{\rm max}$ to $x_{\rm f} + \kappa X_{\rm max}$.

Making the change of variables (35), from which we get

$$\frac{d}{dx} = \frac{1}{\kappa}\frac{d}{dX}, \tag{36}$$

and treating κX as small compared with $x_{\rm f}$, we find that (1) becomes

$$\frac{1}{\kappa}\mathcal{Y}'(X) = -\lambda x_{\rm f}^{-n-2}\left[\mathcal{Y}^2(X) - A^2 x_{\rm f}^3 e^{-2x_{\rm f}}\right], \tag{37}$$

where $\mathcal{Y}(X) = Y(x)$. A consistent dominant balance in this equation is achieved if we take

$$\kappa = x_{\rm f}^{n+2}/\lambda, \tag{38}$$

and if we make this choice, we must neglect the second term on the right side because it is of order λ^{-2} compared with the first term on the right side. This gives the simple inner differential equation

$$\mathcal{Y}'(X) = -\mathcal{Y}^2(X), \tag{39}$$

whose solution is

$$\mathcal{Y}(X) = \frac{1}{X+D}, \tag{40}$$

where D is an integration constant.

To complete the boundary-layer analysis, we must match the two outer solutions to this boundary-layer solution. In order to perform the asymptotic match, we re-express the outer solutions in terms of the inner variable X and then carry out an asymptotic approximation valid for small κ to these asymptotic approximations.

Let us look first at the outer solution in the post-freeze-out region:

$$Y^{\rm post-freeze-out}(X) \sim \frac{1}{1/C - \lambda\left(x_{\rm f} + \kappa X\right)^{-n-1}/(n+1)}, \tag{41}$$

which simplifies to

$$Y^{\rm post-freeze-out}(X) \sim \frac{1}{X + \frac{1}{C} - \frac{\lambda}{(n+1)(x_{\rm f})^{n+1}}}. \tag{42}$$

The coefficient of X in the denominator is 1, which agrees exactly with the coefficient of X in the inner solution (40). Thus, we have achieved an asymptotic match, and the matching condition relates the constants C and D:

$$D = \frac{1}{C} - \frac{\lambda}{(n+1)x_{\rm f}^{n+1}}. \tag{43}$$

Next, we match the boundary-layer solution in (40) to the outer solution (25) in the thermal-equilibrium region. To do so, we must re-express the outer solution in (25) in terms of the inner variable X. Although we are matching to just one term of the inner freeze-out solution, it is essential that we take the first *two* terms in the outer thermal-equilibrium series, and not just the first term, because we have shown that as we approach the freeze-out region, the first two terms in the outer solution become comparable in size. Thus, we include a factor of two in the asymptotic behavior

$$\begin{aligned} Y^{\rm thermal-equilibrium}(x) &\sim 2Ax^{3/2}e^{-x} \\ &\sim 2A\left(x_{\rm f}+\kappa X\right)^{3/2}e^{-x_{\rm f}}e^{-\kappa X} \\ &\sim \frac{1}{X+\frac{\lambda}{x_{\rm f}^{n+2}}}. \end{aligned} \tag{44}$$

Because the coefficient of X in this behavior is 1, we obtain once again a perfect asymptotic match to the inner freeze-out solution in (40). This allows us to determine the value of the constant D:

$$D = \lambda x_{\rm f}^{-n-2}. \tag{45}$$

Finally, combining this result with (43), we obtain the value of C:

$$C = \frac{(n+1)x_{\rm f}^{n+2}}{\lambda\left(n+1+x_{\rm f}\right)}, \tag{46}$$

which is our result for the thermal-relic abundance. For $x_{\rm f}$ large compared with $n+1$ this is in close agreement with the value $(n+1)x_{\rm f}^{n+1}/\lambda$ given in Ref. [8].

B. Boundary-layer analysis of (3)

The arguments given in Subsec. III A apply to a modified version of (3). We modify (3) as follows. If we let $\varphi = |\Phi_0|$, then the substitution

$$Z(x) = Y(x)x^{\varphi}$$

reduces (3) to the simpler Riccati equation

$$Z'(x) = -\lambda x^{-n-2}\left[x^{-\varphi}Z^2(x) - x^{\varphi}Y_{\rm eq}^2(x)\right]. \tag{47}$$

The advantage of this equation over (3) is that there are only three rather than four terms, and thus it is easier to identify a dominant balance.

We can now analyze (47) using the procedure adopted in the previous subsection. In the left outer region (the thermal-equilibrium region) we have

$$Z_0^{\rm thermal-equilibrium}(x) \sim Ae^{-x}x^{\varphi+3/2} \quad \text{and} \quad Z_1^{\rm thermal-equilibrium}(x) \sim \frac{1}{2}x^{\varphi+n+2}. \tag{48}$$

From this result we deduce that the freeze-out value $x_{\rm f}$ is given by

$$x_{\rm f} \sim \log(2A\lambda) - (n+1/2)\log\left(x_{\rm f}\right),$$

which is identical to the result in (30). This result shows that to leading order in $1/\lambda$ the freeze-out temperature is independent of Φ_0; that is, the location of the freeze-out region is only weakly affected by the presence of a dilaton.

Next we discuss the right outer region (post-freeze region). The analog of (34) is

$$Z^{\rm post-freeze-out}(x) \sim \frac{1}{1/C - \lambda x^{-n-1-\varphi}/(n+1+\varphi)}, \tag{49}$$

where C is a constant of integration to be determined by asymptotic matching. As before, C describes the long-term-abundance behavior. However, when x is large compared with the freeze-out temperature ($x \gg x_{\rm f}$), $Y(x)$ does not approach a constant. Rather,

$$Y(x) \sim x^{-\varphi} Z(x) \sim x^{-\varphi} C \quad (x \to \infty). \tag{50}$$

In the freeze-out boundary-layer region we again make the change of variable in (35),

$$x = x_{\rm f} + \kappa X,$$

where the inner variable X may become large compared to 1, but it is still small compared with λ. Thus, since κ is expected to be a small parameter of order $1/\lambda$, the boundary layer is narrow as before. A consistent dominant-balance gives the value

$$\kappa = x_{\rm f}^{n+2+\varphi}/\lambda. \tag{51}$$

The inner differential equation then has the form

$$\mathcal{Z}'(X) = -\mathcal{Z}^2(X), \tag{52}$$

where $\mathcal{Z}(X) = Z(x)$. The solution to (52) is

$$\mathcal{Z}(X) = \frac{1}{X + D}, \tag{53}$$

where D is an integration constant. This is the analog of (40).

An asymptotic match of the right outer solution to the boundary-layer solution produces the relation between the constants C and D,

$$D = \frac{1}{C} - \frac{\lambda}{(n+1+\varphi) x_{\rm f}^{n+1+\varphi}}, \tag{54}$$

which is the analog of (43). Finally, by matching the left outer solution to the boundary-layer solution, we obtain the value of C:

$$C = \frac{(n+1+\varphi) x_{\rm f}^{n+2+\varphi}}{\lambda\,(n+1+\varphi+x_{\rm f})}. \tag{55}$$

In conclusion, we find that, due to the presence of a dilation, the thermal-relic abundance in (50) remains *time dependent*; it vanishes as $x \to \infty$ and does not approach a constant. Note also that if we eliminate the effect of the dilaton by allowing Φ_0 to approach 0, the results in (50) and (55) smoothly reduce to that in Subsec. III A).

IV. APPLICATION OF THE DELTA EXPANSION TO (1) AND (3)

In this section we show how to apply the delta expansion to (1) and (3). We begin with a brief summary of the delta-expansion technique.

A. Summary of the delta expansion

The delta expansion is an unconventional perturbative technique for solving nonlinear problems. It was first introduced to treat nonlinear aspects of quantum field theory [17]. To prepare for applying it to the Boltzmann equations (1) and (3), in this subsection we give a brief review of the delta expansion.

The theme of the delta expansion is to introduce a parameter δ as a measure of the nonlinearity of a problem; that is, the departure of the problem from a corresponding linear problem. We then treat δ as small ($\delta \ll 1$), and solve the problem perturbatively by expanding about the linear problem obtained by setting $\delta = 0$. The basic ideas of the delta expansion are explained in Ref. [11].

To illustrate the delta expansion, we consider the Thomas-Fermi nonlinear boundary-value problem

$$y''(x) = [y(x)]^{3/2}/\sqrt{x}, \qquad y(0) = 1,\ y(+\infty) = 0. \tag{56}$$

This problem is extremely difficult and no closed-form analytical solution is known. We introduce the parameter δ in the exponent of the nonlinear term of the differential equation and consider the one-parameter family of problems

$$y''(x) = y(x)[y(x)/x]^{\delta}, \qquad y(0) = 1,\ y(+\infty) = 0, \tag{57}$$

where we treat δ as a small perturbation parameter. The solution to the unperturbed ($\delta = 0$) linear problem is $y_0(x) = e^{-x}$, and we use $y_0(x)$ as the first term in the delta expansion of the solution to the nonlinear problem (57):

$$y(x) = \sum_{k=0}^{\infty} \delta^k y_k(x). \tag{58}$$

Finally, we recover the solution to the original Thomas-Fermi problem by setting $\delta = 1/2$. Typically, only very few terms are needed in the delta expansion to recover accurate numerical results. Furthermore, the accuracy of the delta expansion can by accelerated by using Padé techniques to sum the delta expansion. In the case of the Thomas-Fermi problem a $(2,1)$-Padé approximant has a numerical error of about 1%.

As a second example, consider the quintic polynomial equation

$$x^5 + x - 1 = 0,$$

which cannot be solved by quadrature. The real root of this equation is $x = 0.75487767\ldots$. Introducing the perturbation parameter δ, we obtain the equation

$$x^{1+\delta} + x = 1.$$

We then seek a perturbation series of the form

$$x(\delta) = c_0 + c_1\delta + c_2\delta^2 + c_3\delta^3 + \ldots \tag{59}$$

whose first term is $c_0 = 1/2$. The radius of convergence of the delta series (59) is 1, and therefore it diverges at $\delta = 4$. However, a $(3,3)$-Padé approximant has a numerical error of 0.05% and a $(6,6)$-Padé approximant has a numerical error of 0.00015%.

B. Delta expansion for (1)

To apply the delta expansion to (1), we insert the parameter δ in such a way that when $\delta = 1$ we recover (1):

$$Y'(x) = -\lambda x^{-n-2}\left(Y - Y_{\rm eq}\right)\left(Y + Y_{\rm eq}\right)^{\delta}. \tag{60}$$

There are, of course, many ways to insert the parameter δ, but the advantage of (60) is that the solution to the unperturbed linear problem obtained by setting $\delta = 0$ is qualitatively similar to the solution to (1), which we have already investigated in Sec. III. In particular, when $\delta = 0$, $Y(x)$ behaves like $Y_{\rm eq}(x)$ for small x, undergoes a transition as x increases, and then approaches a constant as $x \to \infty$.

Following the usual delta-expansion procedure, we represent $Y(x)$ as a series in powers of δ,

$$Y(x) = \sum_{k=0}^{\infty} y_k(x)\delta^k,$$

and then substitute this series into (60). Comparing powers of δ, we obtain a sequence of *inhomogeneous* differential equations for y_k:

$$y_k'(x) + \lambda x^{-n-2} y_k(x) = h_k(x) \quad (k = 0,\, 1,\, 2,\, \ldots), \tag{61}$$

where

$$\begin{aligned}
h_0(x) &= \lambda x^{-n-2} Y_{\rm eq}(x),\\
h_1(x) &= \lambda x^{-n-2}\left[Y_{\rm eq}(x) - y_0(x)\right]\log\left[Y_{\rm eq}(x) + y_0(x)\right],\\
h_2(x) &= \lambda x^{-n-2}\left\{ y_1(x)\frac{Y_{\rm eq}(x) - y_0(x)}{Y_{\rm eq}(x) + y_0(x)} + \frac{Y_{\rm eq}(x) - y_0(x)}{2}\log^2\left[Y_{\rm eq}(x) + y_0(x)\right] - y_1(x)\log\left[Y_{\rm eq}(x) + y_0(x)\right]\right\},
\end{aligned} \tag{62}$$

and so on.

The solution to (61), which is obtained by using the integrating factor $\exp\left[-\lambda x^{-n-1}/(n+1)\right]$, has the quadrature form

$$y_k(x) = e^{\lambda x^{-n-1}/(n+1)} \int_0^x ds\, e^{-\lambda x^{-n-1}/(n+1)} h_k(s). \tag{63}$$

Because (61) is a first-order equation, its solution contains one arbitrary constant for each k and this constant is determined by the requirement that $y_k(0)$ be finite. This requirement fixes the lower endpoint of integration to be 0 for all k. Note that if we evaluate the integral in (63), we obtain the results $y_0(0) = Y_{\rm eq}(0) = 2A\zeta(3)$ (see Appendix B), $y_1(0) = y_2(0) = \ldots = 0$. As x increases, $y_0(x)$ remains close to $Y_{\rm eq}(x)$ until x is of order λ.

We can now express the freeze-out value $Y(\infty)$ as a series in powers of δ and then evaluate this series at $\delta = 1$. Here, we just calculate the first term in ths series:

$$y_0(x) = \lambda e^{\lambda x^{-n-1}/(n+1)} \int_0^\infty ds\, s^{-n-2} e^{-\lambda s^{-n-1}/(n+1)} Y_{\rm eq}(s). \tag{64}$$

Let us evaluate this integral assuming that the parameter λ is large. Since the integrand is exponentially small for small s, we may assume that the only contribution to the integral comes from the region $s \gg 1$, and in this region we may replace $Y_{\rm eq}(s)$ by its asymptotic behavior $As^{3/2}e^{-s}$ (see Appendix A). We thus obtain

$$y_0(\infty) \sim A\lambda \int_0^\infty ds\, s^{-n-1/2} e^{\phi(s)} \quad (\lambda \to \infty), \tag{65}$$

where

$$\phi(s) = -s - \frac{\lambda}{n+1} s^{-n-1}.$$

To evaluate (65) we use Laplace's method with a moving maximum [16]. We note that the maximum of $\phi(s)$, which occurs when $\phi'(s) = 0$, is at $s_0 = \lambda^{1/(n+2)}$. Hence, we introduce the rescaled variable t:

$$s = t\lambda^{1/(n+2)}.$$

This gives the integral

$$y_0(\infty) \sim A\lambda^{5/(2n+4)} \int_0^\infty dt\, t^{-n-1/2} e^{\lambda^{1/(n+2)}\theta(t)} \quad (\lambda \to \infty), \tag{66}$$

where

$$\theta(t) = -t - \frac{1}{n+1} t^{-n-1}.$$

The maximum of $\theta(t)$ occurs at $t = 1$, and near this point we have the quadratic approximation

$$\theta(t) \sim -\frac{n+2}{n+1} - \frac{n+2}{2}(t-1)^2.$$

Thus, evaluating the Gaussian integral, we obtain the result

$$y_0(\infty) \sim \frac{A\sqrt{2\pi}}{\sqrt{n+2}} \lambda^{2/(n+2)} \exp\left[-\frac{n+2}{n+1}\lambda^{1/(n+2)}\right], \tag{67}$$

which reduces to

$$y_0(\infty) \sim A\lambda\sqrt{\pi} e^{-2\sqrt{\lambda}} \tag{68}$$

when $n = 0$. Thus, the delta expansion predicts that at $x = \infty$ the freeze-out value of $Y(x)$ is exponentially small.

We see from this calculation that the delta expansion gives a simple and qualitatively accurate picture of the solution to the Boltzmann equation (1). However, the prediction in (67) of the relic abundance Y_∞ is clearly too small and, of course, this is because we have only kept the leading-order term in the delta expansion. We will see in the next subsection that if we retain higher powers of δ, the qualitative features of the solution do not change but the quantitative prediction for the long-time behavior of $Y(x)$ is improved.

C. Delta expansion for (3)

The delta expansion treatment of (3) parallels that for (1). We insert the parameter δ into (3) as follows:

$$Y'(x) = -\frac{\lambda}{x^{n+2}}[Y(x) - Y_{\rm eq}(x)][Y(x) + Y_{\rm eq}(x)]^{\delta} - \frac{\phi}{x}Y(x), \tag{69}$$

where $\phi = |\Phi_0|$. The analog of (61) is then

$$y_k'(x) + \left(\frac{\lambda}{x^{n+2}} + \frac{\phi}{x}\right) y_k(x) = h_k(x) \quad (k = 0,\, 1,\, 2,\, \ldots). \tag{70}$$

The solution to (70), which is obtained by using the integrating factor $x^{\phi} \exp\left[-\lambda x^{-n-1}/(n+1)\right]$, has the quadrature form

$$y_k(x) = x^{-\phi} \exp\left[\lambda x^{-n-1}/(n+1)\right] \int_0^x ds\, s^{\phi} \exp\left[-\lambda s^{-n-1}/(n+1)\right] h_k(s). \tag{71}$$

Using the modified Laplace method again, we obtain for $x \to \infty$ and large λ the asymptotic approximation

$$y_0(x) \sim x^{-\phi} B(\lambda), \tag{72}$$

where the constant $B(\lambda)$ is given by

$$B(\lambda) = A\lambda^{(\phi+2)/(n+2)} \sqrt{\frac{2\pi}{n+2}} \exp\left[-\lambda^{1/(n+2)}(n+2)/(n+1)\right].$$

This shows that the dilatonic correction to the Boltzmann equation gives a significant qualitative change in the freeze-out behavior of DM. The magnitude of the DM abundance is era dependent because its leading behavior for large x is an algebraic decay of the form $x^{-\phi}$. The delta expansion is qualitatively in agreement with boundary layer theory.

The result in (72) is the analog of (67), and again we see that while the delta expansion in leading-order gives a good qualitative description of the solution to the Boltzmann equation, the quantitative prediction for the coefficient $B(\lambda)$ of $x^{-\phi}$ in the large-x behavior is much too small. Thus, we extend the result in (72) to first order in δ. The calculation is a straightforward generalization of the zeroth-order calculation and the result is

$$y_0(x) + \delta y_1(x) \sim x^{-\phi} B(\lambda) \left\{1 - \delta \log[B(\lambda)] + \delta \frac{\phi}{n+1}\left[\gamma + \log\left(\frac{\lambda}{n+1}\right)\right]\right\}, \tag{73}$$

where $\gamma = 0.5772\ldots$ is Euler's constant.

For large λ, we can ignore all but the $\log[B(\lambda)]$ term, and we obtain a rough asymptotic behavior, which is a simplified version of (73):

$$y_0(x) + \delta y_1(x) \sim x^{-\phi} B(\lambda) \left\{1 - \delta \log[B(\lambda)]\right\}. \tag{74}$$

Not surprisingly, the second-order contribution contains a logarithm squared:

$$y_0(x) + \delta y_1(x) \sim x^{-\phi} B(\lambda) \left\{1 - \delta \log[B(\lambda)] + \frac{1}{2}\delta^2 \log^2[B(\lambda)]\right\}. \tag{75}$$

In general, the dominant contribution to the coefficient of δ^k in the delta expansion is $(-1)^k \log^k[B(\lambda)]/k!$. Thus, if we sum the approximate delta series to all orders in δ and set $\delta = 1$, the multiplicative coefficient $B(\lambda)$, which is numerically incorrect because it is much too small, is exactly canceled. This explains the mechanism by which the delta expansion and the matched asymptotic analysis become compatible.

V. BRIEF CONCLUDING REMARKS

We have applied two powerful perturbative techniques, boundary-layer theory and the delta expansion, to find globally accurate solutions to two different Boltzmann equations that describe dark-matter abundances in the early universe. The first Boltzmann equation is based on the standard model of particle physics and general relativity; the second includes additional effects due to dilatonic contributions that arise in string theory. The boundary-layer

solution consists of contributions from three distinct eras, a thermal-equilibrium epoch, a freeze-out region, and a nonequilibrium relic-abundance epoch, and the global solution is obtained by the use of asymptotic matching. The delta-expansion solution does not require the use of asymptotic matching and gives a good qualitative picture of the behavior in these three epochs, but the results to low orders in δ are not as accurate for long times.

We have shown that when dilatonic effects are not included, the dark-matter-relic abundance approaches a constant for long times, but when dilatonic effects are included, the relic abundance has a power-law decay determined by the dilaton coupling.

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Appendix A: Large-x behavior of $Y_{\rm eq}(x)$

In this Appendix we derive the large-x asymptotic behavior of the equilibrium distribution $Y_{\rm eq}$ in (2), whose integral representation is given by

$$Y_{\rm eq}(x)=A\int_{s=0}^\infty ds\frac{s^2}{e^{\sqrt{s^2+x^2}}-1}. \tag{A1}$$

When $x\gg 1$, we can neglect -1 in denominator of the integrand to all orders in powers of $1/x$ and write

$$Y_{\rm eq}(x)\sim A\int_0^\infty ds\,s^2e^{-\sqrt{s^2+x^2}}\quad(x\to\infty). \tag{A2}$$

The scaling $s=xt$ followed by the change of variables $u=\sqrt{t^2+1}$ then gives the integral representation

$$Y_{\rm eq}(x)\sim Ax^3\int_{u=1}^\infty du\,e^{-xu}u\sqrt{u^2-1}\quad(x\to\infty). \tag{A3}$$

Watson's lemma [16] applies directly to the integral (A3). The procedure is first to expand $u\sqrt{u^2+1}$ as a series in powers of $u-1$,

$$u\sqrt{u^2-1}=\sum_{n=0}^\infty a_n(u-1)^{n+1/2}, \tag{A4}$$

where

$$a_n=\frac{1}{\sqrt{2\pi}}(-1/2)^n\frac{(n+3/2)\Gamma(n-3/2)}{n!}, \tag{A5}$$

and then to interchange orders of summation and integration. Integrating term by term gives the asymptotic series

$$Y_{\rm eq}(x)\sim Ae^{-x}x^{3/2}\frac{1}{\sqrt{2\pi}}\sum_{n=0}^\infty\frac{\Gamma(n+5/2)\Gamma(n-3/2)}{(-2x)^n n!}\quad(x\to\infty). \tag{A6}$$

Thus, the series begins

$$Y_{\rm eq}(x)\sim Ae^{-x}x^{3/2}\sqrt{\frac{\pi}{2}}\left(1+\frac{15}{8x}+\dots\right)\quad(x\to\infty). \tag{A7}$$

Appendix B: Small-x behavior of $Y_{\rm eq}(x)$

In this appendix we show how to find the small-x asymptotic behavior of the integral

$$Y_{\rm eq}(x) = A\int_{s=0}^{\infty} ds\frac{s^2}{e^{\sqrt{s^2+x^2}}-1}. \tag{B1}$$

We begin by substituting $t=\sqrt{s^2+x^2}$. This gives

$$Y_{\rm eq}(x) = A\int_{t=x}^{\infty} dt\frac{t\sqrt{t^2-x^2}}{e^t-1} = A\int_{t=x}^{\infty} dt\frac{t^2}{e^t-1}\left(\sqrt{1-x^2/t^2}-1+1\right) = \mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}, \tag{B2}$$

where

$$\begin{aligned}
\mathcal{A} &= Y_{\rm eq}(0) = A\int_{t=0}^{\infty} dt\frac{t^2}{e^t-1} = 2A\zeta(3),\\
\mathcal{B} &= -A\int_{t=0}^{x} dt\frac{t^2}{e^t-1},\\
\mathcal{C} &= A\int_{t=1}^{\infty} dt\frac{t^2}{e^t-1}\left(\sqrt{1-x^2/t^2}-1\right),\\
\mathcal{D} &= A\int_{t=x}^{1} dt\frac{t^2}{e^t-1}\left(\sqrt{1-x^2/t^2}-1\right).
\end{aligned} \tag{B3}$$

We now evaluate each of the integrals $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, in turn.

To evaluate $\mathcal{B}$ we expand $t/(e^t-1)$ in a Taylor series, which converges if $|t|<2\pi$, and integrate term by term:

$$\mathcal{B} = -A\int_0^x dt\, t\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n = -A\sum_{n=0}^{\infty}\frac{B_n}{(n+2)n!}x^{n+2}, \tag{B4}$$

where B_n is the nth Bernoulli number ($B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, …, $B_{2n+1}=0$ for $n\geq 1$). So,

$$\mathcal{B} = -\frac{A}{2}x^2+\frac{A}{6}x^3-\frac{A}{48}x^4+\ldots. \tag{B5}$$

To evaluate $\mathcal{C}$ and $\mathcal{D}$ we use the expansion

$$\sqrt{1-a}-1 = -\frac{1}{2\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma(n-1/2)}{n!}a^n. \tag{B6}$$

Thus, $\mathcal{C}$ becomes

$$\mathcal{C} = -\frac{A}{2\sqrt{\pi}}\int_1^{\infty} dt\frac{t^2}{e^t-1}\sum_{n=1}^{\infty}\frac{\Gamma(n-1/2)}{n!}x^{2n}t^{-2n} = -\frac{A}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{(n+1)!}x^{2n+2}\int_1^{\infty} dt\frac{t^{-2n}}{e^t-1}. \tag{B7}$$

Hence,

$$\mathcal{C} = c_2x^2+c_4x^4+{\rm O}\left(x^6\right), \tag{B8}$$

where

$$c_2 = -\frac{1}{2}\int_1^{\infty}\frac{dt}{e^t-1} = \frac{1}{2}\log(1-1/e) \qquad \text{and} \qquad c_4 = -\frac{1}{8}\int_1^{\infty}\frac{dt}{t^2(e^t-1)}. \tag{B9}$$

The interesting contribution comes from D. We express D as the double sum

$$\mathcal{D} = -\frac{1}{2\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{B_m}{m!}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{(n+1)!}x^{2n+2}\int_{t=x}^{1} dt\, t^{m-1-2n}. \tag{B10}$$

Depending on the values of m and n in the sum, we get different kinds of terms. For example, logarithm terms appear when (and only when) $m = 2n$. Thus, all the logarithm terms appear in the series

$$\mathcal{D}_{\text{log terms}} = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{B_{2n}\Gamma(n+1/2)}{(n+1)!(2n)!} x^{2n+2} \log x = \frac{1}{2}x^2 \log x - \frac{1}{48}x^4 \log x + \ldots . \tag{B11}$$

Terms of order x^2 arise from the upper endpoint of integration in (B10) when $n = 0$ and for all $m \geq 1$ (but not $m = 0$ because this gives rise to a log term, and we have already included this contribution) and they arise from the lower endpoint of integration when $m = 0$ for all $n \geq 1$ (but not $n = 0$). The upper endpoint gives

$$\mathcal{D}_{\text{upper, 2}} = -\frac{1}{2}x^2 \int_{t=0}^{1} dt \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) = -\frac{1}{2} \log(1 - 1/e) x^2. \tag{B12}$$

The lower endpoint gives

$$\mathcal{D}_{\text{lower, 2}} = x^2 \int_{t=0}^{1} \frac{dt}{t^3} \left(\sqrt{1 - \frac{1}{2}t^2} + t^2 - 1 \right) = \frac{1}{4}x^2 - \frac{1}{2}\log(2) x^2. \tag{B13}$$

Thus, the result for the expansion of $I_{\text{eq}}(x)$ in (B1) to order x^2 is

$$I(x) \sim 2\zeta(3) + \left[\frac{1}{2}\log(x/2) - \frac{1}{4} \right] x^2 + \ldots . \tag{B14}$$

We have pursued this calculation to higher order in powers of x, and we find that in (B14) the coefficients of x^3 and x^5 are 0, the coefficient of x^4 is

$$\frac{\gamma}{96} + \frac{\zeta'(-1)}{8} - \frac{1}{128} - \frac{\log(2)}{96} + \frac{\log(x)}{96} = -0.0297 + 0.0104 \log(x), \tag{B15}$$

and the coefficient of x^6 is

$$\frac{\gamma}{192} + \frac{\zeta'(-1)}{16} + \frac{979}{268800} + \frac{\pi}{2880} - \frac{\log(x)}{11520} = -0.0048 + 0.0000868 \log(x), \tag{B16}$$

where $\gamma = 0.57721$ is Euler's constant.

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